A RESIDUAL CORRECTION METHOD FOR ITERATIVE RECONSTRUCTION WITH INACCURATE SYSTEM MODEL

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ABSTRACT

The quality of images produced by iterative image reconstruction methods is directly affected by the accuracy of the system model being used. Although the system response of an imaging system can be computed or measured with high accuracy, practical constraints on computation cost often force the adoption of various approximations to the system model to obtain a computationally efficient system matrix. These approximations inevitably cause artifacts in reconstructed image. In this work we propose a residual correction method for iterative reconstruction to reduce reconstruction artifact caused by the model inaccuracies. Unlike conventional iterative methods which assume that the system matrix is accurate, the proposed method reconstructs an initial image with an approximate system model and then corrects for the reconstruction artifacts by introducing a data correction term to compensate for the model inaccuracies. Computer simulation showed that the proposed method can significantly improve image quality in terms of objective function value compared to approximate algorithms, while it is computationally more efficient than the conventional method that uses the accurate system model at every iteration.

***Index Terms*—** Iterative reconstruction, system matrix, error propagation, artifact correction.

1. INTRODUCTION

Iterative reconstruction methods have been developed for many medical imaging modalities because they can incorporate sophisticated and accurate system models. Several groups have demonstrated that an accurate system matrix can improve image quality in both SPECT and PET. Although Monte Carlo simulation or physical measurements can be used to obtain a fairly accurate system model [1], the resulting models are often non-sparse and using them in a conventional iterative reconstruction framework for emission tomography is computationally challenging, because of the high computational cost associated with the forward and back projections with a non-sparse system matrix required at every iterations. One method to reduce computational cost is to use a simplified unmatched back projector [2], but an accurate system model for forward projection is still required at every iteration.

For efficient computation, most iterative reconstruction methods employ approximations to obtain a sparse system matrix or a system matrix with a certain structure (e.g. block circulant). For example, most system models of PET ignore the effects of positron range, acollinearity of photon pairs, variation of sensitivity along the line of response (LOR) caused by the solid angle effect, and the spatially variant response of the detector caused by the inter-crystal penetration and scatter. Approximations in system models are also necessary when graphical processing unit (GPU) or Fourier projectors are used to accelerate reconstruction [3, 4].

Using approximate system model inevitably affects image quality. The inaccuracy in system model is one of the major factors that limit the resolution in high-count PET studies. With the advent of new generation of high resolution detectors, there is increasing demands for reducing artifacts caused by errors in system matrix.

Based on our previous work on error propagation from system matrix in the maximum *a posteriori* (MAP) reconstruction [5], here we propose a residual correction approach that is capable of reducing reconstruction artifacts caused by errors in the system model. Unlike convention iterative methods, the proposed method does not require an accurate system model at every iteration. The only knowledge needed for error correction is a forward projection of the image with the accurate system model, and no backprojection is required. In this way, we are able to improve image resolution by using the accurate system model yet retain the speed of approximate system model.

2. THEORY

2.1. MAP Image reconstruction

Emission data $\mathbf{y} \in \mathbb{R}^{M\times 1}$ are related to the unknown tracer distribution $\mathbf{x} \in \mathbb{R}^{N\times 1}$ through an affine transform

$$\mathbf{y} = \mathbf{P}\mathbf{x} + \mathbf{r} + \mathbf{n},$$

where $\mathbf{P} \in \mathbb{R}^{M\times N}$ is the system matrix; $\mathbf{r} \in \mathbb{R}^{M\times 1}$ accounts for the presence of scattered and random events in the data, and $\mathbf{n} \in \mathbb{R}^{M\times 1}$ is zero-mean random noise.

Estimation $\hat{\mathbf{x}}$ is found by maximizing the log-posterior density function $\Psi(\mathbf{y},\mathbf{x})$

$$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}\geq 0} \Psi(\mathbf{y},\mathbf{x}), \quad \Psi(\mathbf{y},\mathbf{x}) = [L(\mathbf{y}\,|\,\mathbf{x}) - \beta U(\mathbf{x})], \qquad (1)$$

where $L(\mathbf{y}\,|\,\mathbf{x})$ is the log-likelihood function, $U(\mathbf{x})$ is a roughness penalty function, and β is the regularization parameter that controls the tradeoff between resolution and noise.

For Gaussian noise, we have $L(\mathbf{y}\,|\,\mathbf{x}) = -\frac{1}{2}\|\mathbf{y}-\hat{\mathbf{y}}\|_{\mathbf{w}}^2$, where $\mathbf{w}$ is the inverse of the covariance matrix of $\mathbf{y}$ and $\hat{\mathbf{y}} \equiv \mathbf{P}\hat{\mathbf{x}} + \mathbf{r}$. For independent Poisson noise, we have $L(\mathbf{y}\,|\,\mathbf{x}) = \sum_i (y_i \log \hat{y}_i - \hat{y}_i - \log y_i!)$.

2.2. Error propagation from the system matrix

Iterative algorithms are often used to find the MAP solution in (1) because a closed-form solution does not usually exist. Even when the closed-form solution is available, such as in the case of Gaussian noise, the solution is often impractical to compute for real imaging systems because of the high dimension. While iterative methods are amenable to arbitrarily complicated system models, approximations are intentionally introduced to the system matrix $\mathbf{P}$ to allow fast forward and back projection operations. Consider error $\Delta\mathbf{P}$ in a system model, i.e.,

$$\mathbf{P}_{true} = \mathbf{P} - \Delta\mathbf{P},$$

where $\mathbf{P}_{true}$ is the "accurate" system matrix, such as one calculated by a validated Monte Carlo simulation program, and $\mathbf{P}$ is the approximate system matrix, such as an ideal line integral system model used in many reconstruction programs. The corresponding error propagation from $\Delta\mathbf{P}$ to the estimated image is

$$\Delta\hat{\mathbf{x}} \equiv \hat{\mathbf{x}}^* - \hat{\mathbf{x}},$$

where $\hat{\mathbf{x}}^*$ and $\hat{\mathbf{x}}$ are the images reconstructed using the true and approximate system matrices, respectively.

In [5] we analyzed the effect of errors in the system matrix on MAP reconstruction using Kuhn-Tucker condition and the first order Taylor series approximations. For any log-likelihood term that can be written as

$$L(\mathbf{y};\mathbf{x}) = \Phi(\mathbf{y},\hat{\mathbf{y}}),$$

the error propagation formula is

$$\Delta\hat{\mathbf{x}} \approx \left[-\mathbf{P}^T\nabla^{02}\Phi(\mathbf{y},\hat{\mathbf{y}})\mathbf{P} + \beta\nabla^2 U(\hat{\mathbf{x}})\right]^{-1} \times\left[(\Delta\mathbf{P})^T\nabla^{01}\Phi(\mathbf{y},\hat{\mathbf{y}}) + \mathbf{P}^T\nabla^{02}\Phi(\mathbf{y},\hat{\mathbf{y}})(\Delta\mathbf{P})\hat{\mathbf{x}}\right],$$

where "^{T}" denotes matrix transpose, the (i,j)th element of $\nabla^{02}\Phi(\mathbf{y},\hat{\mathbf{y}})$ is $\frac{\partial^2}{\partial\hat{y}_j\partial\hat{y}_k}\Phi(\mathbf{y},\hat{\mathbf{y}})$, the jth element of $\nabla^{01}\Phi(\mathbf{y},\hat{\mathbf{y}})$ is $\frac{\partial}{\partial\hat{y}_j}\Phi(\mathbf{y},\hat{\mathbf{y}})$, and $\nabla^2 U(\hat{\mathbf{x}})$ is a symmetric matrix with the (i,m)th element being $\frac{\partial^2}{\partial\hat{x}_j\partial\hat{x}_m}U(\hat{\mathbf{x}})$.

For Gaussian likelihood function, the expression can be simplified to

$$\Delta\hat{\mathbf{x}} \approx -\left[\mathbf{P}^T\mathbf{w}\mathbf{P} + \beta\nabla^2 U(\hat{\mathbf{x}})\right]^{-1}\left[\mathbf{P}^T\mathbf{w}(\Delta\mathbf{P})\hat{\mathbf{x}}\right]. \qquad (2a)$$

For Poisson likelihood function, the expression can be simplified to

$$\Delta\hat{\mathbf{x}} \approx -\left[\mathbf{P}^T D\left[\frac{y}{\hat{y}^2}\right]\mathbf{P} + \beta\nabla^2 U(\hat{\mathbf{x}})\right]^{-1}\left[\mathbf{P}^T D\left[\frac{y}{\hat{y}^2}\right](\Delta\mathbf{P})\hat{\mathbf{x}}\right], \qquad (2b)$$

where $D\left[\frac{y}{\hat{y}^2}\right]$ is a diagonal matrix whose ith diagonal element equals to $y_i/\hat{y}_i^2$.

2.3. Correction with pre-compensation of data

We found that equation (2a) and (2b) can also be expressed as

$$\Delta\hat{\mathbf{x}} \approx \nabla_{\mathbf{y}}\hat{\mathbf{x}}(\mathbf{y})(\Delta\mathbf{P})\hat{\mathbf{x}}, \qquad (3)$$

where $\hat{\mathbf{x}}(\mathbf{y})$ is the reconstruction function implicitly defined by (1), and $\nabla_{\mathbf{y}}\hat{\mathbf{x}}(\mathbf{y})$ is the gradient of $\hat{\mathbf{x}}$ with respect to $\mathbf{y}$.

Using (3) we can obtain an error-corrected image by

$$\begin{aligned}\hat{\mathbf{x}}^{new} &= \hat{\mathbf{x}} + \nabla_{\mathbf{y}}\hat{\mathbf{x}}(\mathbf{y})(\Delta\mathbf{P})\hat{\mathbf{x}} \\ &\approx \arg\max_{\mathbf{x}\geq 0}\Psi(\mathbf{y}+(\Delta\mathbf{P})\hat{\mathbf{x}},\mathbf{x})\end{aligned} \qquad (4)$$

where $\Psi(\mathbf{y},\mathbf{x})$ is the log posterior density function defined in (1). Equation (4) indicates that a corrected image can be reconstructed using the approximate system matrix after precompensating the data for the error propagation. The correction term is computed by comparing the forward projection of the initial reconstruction $\hat{\mathbf{x}}$ using the approximate model and the accurate model.

The advantage of correction using (4) is that no separate optimization algorithm is needed. Instead, the original reconstruction program using the approximate matrix can be used for artifact correction. It should be noted that the knowledge required for computing the data correction term is only a forward projection using the true system matrix, not the full system matrix itself. In particular, no backprojection operation using the accurate system model is required.

2.4. Iterative residual correction

While equation (4) indicates that reconstruction artifacts can be removed, the correction is only an approximation because of the first-order Taylor approximation that we used in deriving (2a) and (2b). The corrected image, $\hat{\mathbf{x}}^{new}$, still contains residual errors. To further reduce the reconstruction artifacts, we may consider re-applying correction to $\hat{\mathbf{x}}^{new}$ and repeating this process iteratively. The general procedure for iterative residual correction is

$$\mathbf{x}^{(n+1)} = \arg\max_{\mathbf{x}\geq 0}\Psi(\mathbf{y}+\Delta\mathbf{y}^{(n)},\mathbf{x}) \qquad (5)$$

where

$$\Delta\mathbf{y}^{(n)} = (\Delta\mathbf{P})\hat{\mathbf{x}}^{(n)} \qquad (6)$$

is the data correction term at the nth iteration of correction.

General convergence properties of the proposed iterative correction scheme is still an open problem. Here we present some initial results on the convergence condition for weighted least squares (WLS) reconstruction without regularization and non-negativity constraint on the image. In this case, equation (5) simplifies to

$$\hat{\mathbf{x}}^{(n+1)} = \arg\min_{\mathbf{x}} \left\{ \left\| \mathbf{y} + \Delta\mathbf{y}^{(n)} - \mathbf{P}\mathbf{x} \right\|_{\mathbf{w}}^{2} \right\}.$$

An implicit iterative formula for $\hat{\mathbf{x}}^{(n+1)}$ is defined by the corresponding normal equation

$$(\mathbf{P}^T \mathbf{w}\mathbf{P})\hat{\mathbf{x}}^{(n+1)} = \mathbf{P}^T \mathbf{w}(\mathbf{y} + \Delta\mathbf{y}^{(n)}). \quad (7)$$

Substitute (6) into (7), we have

$$\hat{\mathbf{x}}^{(n+1)} = \hat{\mathbf{x}}^{(n)} + (\mathbf{P}^T \mathbf{w}\mathbf{P})^{-1} \left[\mathbf{P}^T \mathbf{w}\mathbf{y} - \mathbf{P}^T \mathbf{w}\mathbf{P}_{true}\hat{\mathbf{x}}^{(n)} \right]$$

which is a generalized Landweber iteration. The corresponding convergence condition is

$$\rho\left[\mathbf{I} - (\mathbf{P}^T \mathbf{w}\mathbf{P})^{-1}(\mathbf{P}^T \mathbf{w}\mathbf{P}_{true}) \right] < 1 \quad (8)$$

where $\rho\left[\cdot\right]$ is the matrix spectrum radius. If (8) is satisfied, the limit of $\hat{\mathbf{x}}^{(n)}$ is

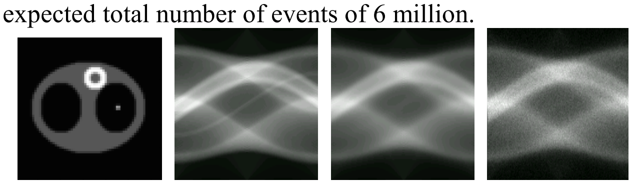
$$\hat{\mathbf{x}}^{(\infty)} = (\mathbf{P}^T \mathbf{w}\mathbf{P}_{true})^T \mathbf{P}^T \mathbf{w}\mathbf{y}.$$

In particular, if $\text{range}(\mathbf{P}) = \text{range}(\mathbf{P}_{true})$, then $\hat{\mathbf{x}}^{(\infty)} = \hat{\mathbf{x}}^*$, which means the proposed iterative correction method will converge to exactly the same result as the conventional method that uses the true system matrix. These results are very similar to that of using unmatched projector and backprojector pairs [2].

3. COMPUTER SIMULATION

To illustrate the advantage of the proposed method, we simulated a 2D imaging system having 180 angular views over 180 degrees and 128 radial samples per view. The image is represented by 64 by 64 square pixels. The width of each line of response is one half of the pixel size.

Fig. 1(a) shows the 2D digital mouse chest phantom [6] used in this study. The simulated projection data is computed by first applying a space shift-variant blurring filter on the object, and then forward projecting the blurred object using the 2D parallel beam Radon transform. The blurring filter was intended to simulate the positron range effect in PET, but it can also be considered as a general image degrading factor in an imaging system. For simplicity, here we assumed that the positrons behave diffusively [7], and the blurring filter was implemented by numerically solving a space variant diffusion PDE (partial differential equation). The relative diffusion constant for air, lungs, and other body tissues were set to 5, 3, and 1, respectively. A more accurate forward projector based on Monte Carlo simulation is to be used in the future. Fig. 1(b) and (c) show noiseless projections of the phantom with and without the blurring filter, respectively. Fig. 1(d) shows the noisy projection data generated by adding computer generated Poisson noise to the image shown in Fig. 1(c), with an expected total number of events of 6 million.



(a) (b) (c) (d)

Fig. 1 (a) The digital phantom. (b) Noiseless projection data, no blurring. (c) Noiseless projection data, with blurring. (d) Noisy projection data, with blurring.

The proposed method aims to reconstruct the phantom image with an approximate system matrix, which is a pure geometric projection matrix without modeling the positron range effect at all. A penalized log-likelihood objective function with a quadratic penalty which penalizes the differences between neighboring pixels (8 nearest neighbors in 2D) was used in reconstruction. Three iterative reconstruction methods were compared: (1) the conventional method that uses the accurate system model in every iteration; (2) the conventional method that uses the approximate system model in every iteration; (3) the proposed method which uses an approximate system model in the reconstruction and uses the accurate system model for artifact correction. We investigated applying the correction every 5, 10, or 20 iterations. A uniform image was used as the initial for all the methods. A preconditioned conjugate gradient algorithm with an EM-type preconditioner was used for calculating the image update in all the methods. Both noiseless and noisy data were studied.

Fig. 2 plots the objective function values as a function of iteration number for the noiseless reconstructions. The proposed method increased the objective function value significantly compared to the conventional method that uses only the approximate system model (method 2). In fact, the proposed method improves the objective function value at a similar rate as the conventional methods that use the accurate system model (method 1) at every iteration. Note that the proposed method is targeted to the situation where a forward/back projection using an approximate system model is far more efficient than using the accurate system model. Because the proposed method does not involve a forward or back projection with the accurate system model at every iteration, the computation cost of the proposed method can be orders of magnitude less than that of the conventional method (method 1).

The results of the noisy data are shown in Fig. 3. Compared with the noiseless reconstructions, the objective function values plateau quickly for all the methods, but the relative performance remains similar to that in Fig. 2. The performance of the proposed method is slightly degraded, because of the noise and also the regularization imposed.

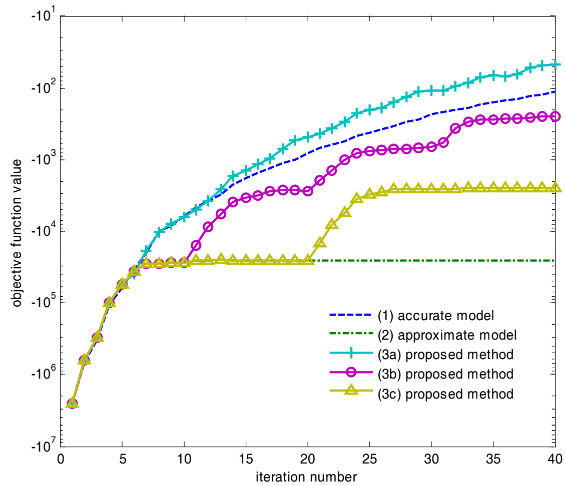


Fig. 2. Objective function values of the noiseless reconstructions as a function of iteration number. No regularization is imposed. (1) the conventional method using the accurate system model; (2) the conventional method using the approximate system model; (3a) (3b) (3c): the proposed method that corrects for the reconstruction artifacts every 5, 10, and 20 iterations, respectively. Note that the computation cost per iteration of each method is different.

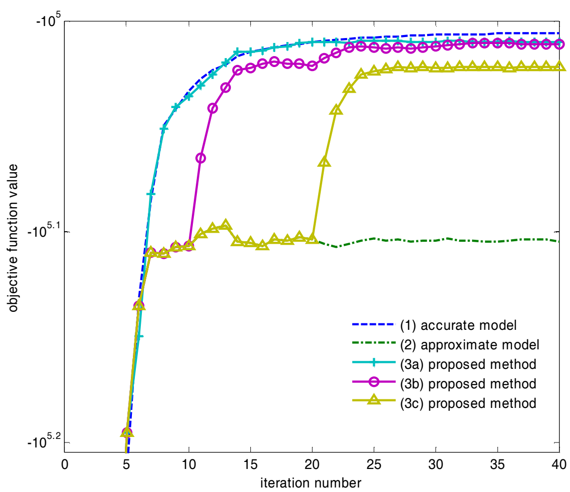


Fig. 3. Objective function values for the noisy reconstructions as a function of iteration number. The legends are the same as those in Fig. 2.

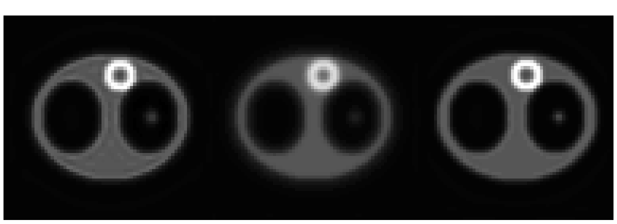


Fig. 4. The images reconstructed from the noiseless data by different methods. No regularization was imposed. The image labels are the same as the legends in Fig. 2. All images were obtained with 40 iterations.

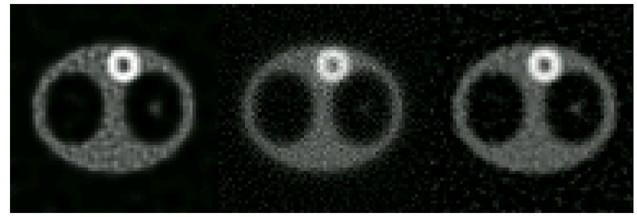


Fig. 5. The images reconstructed from a noisy dataset by different methods. Regularization was imposed to reduce noise. The image labels are the same as the legends in Fig. 2. All images were obtained with 40 iterations.

Fig. 4 and Fig. 5 compare the images reconstructed by the three methods. The images reconstructed by the proposed method are visually similar to those by the conventional method using the accurate system model.

5. CONCLUSION AND DISCUSSION

We have presented a residual correction method for iterative reconstruction with an inaccurate system matrix. The proposed method compensates for the model mismatch by introducing a correction term to the data to cancel out the error between the approximate model and accurate model. The major advantage is that the proposed method does not require an accurate system model at every iteration. Computer simulation showed that with a few correction iterations, the proposed method achieved comparable objective function value to that achieved by the conventional method with tens of iterations. Future work includes studying the possibility of using a hierarchy of approximate system matrix to further accelerate reconstruction and applying the algorithm to real data.

6. REFERENCES

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